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Mass transport in sector and annular sector tubes

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Abstract—Velocity distribution and mass transport for plug and laminar flow are determined for liquid flow in sector and annular sector tubes. The wall concentration and the initial concentration of the inlet to the tube are considered constant. The concentration profiles are determined in various locations of the cross-section and along the length of the tube. In addition lines of equal liquid velocity and equal concentration are presented for the plug and laminar flow case.

1. INTRODUCTION

THE MASS transport problem has been investigated in channels and tubes for plug and laminar flow for many decades [1–6]. In most of these cases the partial differential equation obtained exhibited only two variables, while the velocity distribution could be presented as a simple one-variable function. In a sector or annular sector tube, however, the angular coordinate ϕ appears, such that the diffusion equation exhibits the three coordinates r, ϕ, z, while the liquid velocity inside the tube may only be described by an infinite series, where the terms exhibit functions of the two coordinates r and ϕ. This yields a mass transport equation which does not admit a completely analytical solution. For this reason the plug flow solution is used as an expansion function for the laminar flow case. With the Galerkin method applied to the mass transport problem with laminar flow, the Galerkin condition yields an infinite number of algebraic equations, of which the vanishing of the truncated coefficient determinant renders the approximation of the lower eigenvalues. Since with increasing axial coordinate the exponential function, describing the mass transport behaviour along the tube length, rapidly approaches vanishing magnitude, the convergence of the result presenting the concentration is very good. In the course of the numerical evaluation it was, however, found, that the numerical procedure, based on the analytical solution (i.e. the treatment of the coefficient determinants and the integrals of Bessel functions), needs considerably more computer time than the original numerical solution of the mass transport problem. This numerical solution for both cases, i.e. plug and laminar flow, was in addition used to prove the accuracy of the analytical solution. They produce identical results.

2. BASIC EQUATIONS

For the determination of the local concentration one has to solve the partial differential equation for mass transport. This may be performed for the case of a plug flow and that of laminar flow in a tube of annular sector cross-section.

2.1. *The flow problem*

Assuming laminar flow along the z-axis for which only the axial flow component $w \neq 0$, one has to solve for a pipe with annular sector cross-section of angle $2\pi\alpha$ (Fig. 1) with radial and angular flow velocity $u = v = 0$ and continuity equation $\partial w/\partial z = 0$ for stationary flow the Navier–Stokes equation

$$\frac{1}{\eta}\frac{\partial p}{\partial z} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \phi^2} \tag{1}$$

with the no-slip boundary condition at

$$w = 0 \quad \text{for } \phi = 0, 2\pi\alpha \text{ and } r = a, b. \tag{2}$$

Expanding $w(r,\phi)$ into a Fourier sine series satisfying the boundary conditions at $\phi = 0$ and $2\pi\alpha$ yields

$$w(r,\phi) = \sum_{m=1}^{\infty} W_m(r)\sin\left(\frac{m}{2\alpha}\phi\right). \tag{3}$$

With the expansion of the pressure gradient

$$\frac{1}{\eta}\frac{\partial p}{\partial z} = \sum_{m=1}^{\infty} P_m \sin\left(\frac{m}{2\alpha}\phi\right) \tag{4}$$

with

$$P_m = \begin{cases} 0 \text{ for } m \text{ even} \\ \dfrac{4}{\eta\pi m}\left(\dfrac{\partial p}{\partial z}\right) \text{ for } m \text{ odd} \end{cases} \tag{5}$$

one obtains the differential equation

$$\frac{d^2 W_m}{dr^2} + \frac{1}{r}\frac{dW_m}{dr} - \frac{m^2}{4\alpha^2 r^2}W_m = \begin{cases} 0 \text{ for } m \text{ even} \\ \dfrac{4}{\eta\pi m}\dfrac{\partial p}{\partial z} \text{ for } m \text{ odd} \end{cases} \tag{6}$$

for the determination of the function $W_m(r)$. The solution of this differential equation satisfying the remain-

NOMENCLATURE

a	outer radius of annular sector tube	p	liquid pressure
b	inner radius of annular sector tube	r, ϕ, z	cylindrical polar coordinates
c	concentration	$\dot{V}$	volumetric flow
c_i	initial concentration at inlet $z = 0$	w	axial liquid velocity
c_w	wall concentration at $r = a, b$; $\phi = 0, 2\pi\alpha$	w_0	plug flow velocity.
D	diffusion coefficient		
$J_{m/2\alpha}, Y_{m/2\alpha}$	Bessel functions of first and second kind of order $m/2\alpha$	Greek symbols	
k	b/a	β_{mn}	eigenvalues
		η	dynamic viscosity
		$2\pi\alpha$	sector angle of tube cross-section.

ing boundary conditions $W_m = 0$ at $r = a, b$ is given by ($W_{2m} = 0$)

$$W_{2m-1}(r) = \left(4a^2\left(\frac{\partial p}{\partial z}\right)\right/\eta\pi(2m-1) \times\left[4-\frac{(2m-1)^2}{4\alpha^2}\right]\Bigg)\left\{\left(\frac{r}{a}\right)^2 - [(1-k^{(2m-1)/2\alpha+2})/(1-k^{(2m-1)/\alpha})]\left(\frac{r}{a}\right)^{(2m-1)/2\alpha} - [(k^{(2m-1)/2\alpha}(k^2-k^{(2m-1)/2\alpha})/(1-k^{(2m-1)/\alpha})]\left(\frac{a}{r}\right)^{(2m-1)/2\alpha}\right\}. \quad (7)$$

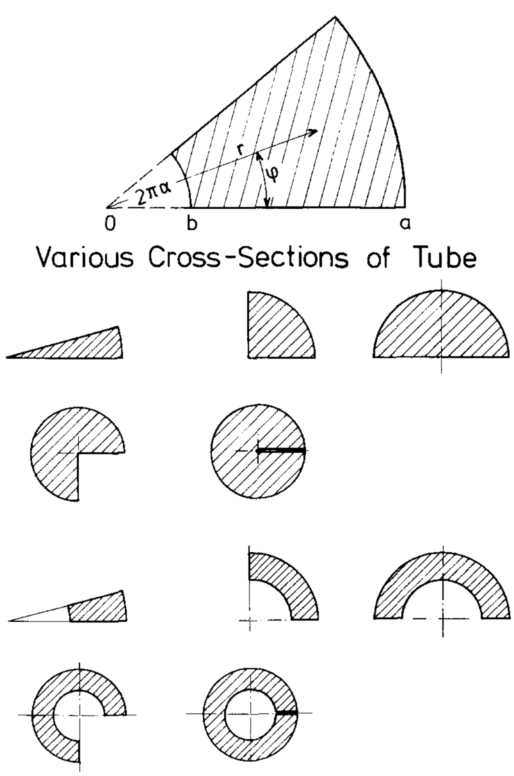


Fig. 1. Geometry of cross-section of tube and coordinate system.

One has to observe, for this solution, that $\alpha \neq 1/4, 3/4$ for $m = 1$ and 2, respectively. For $\alpha = 1/4$ one obtains the solution

$$W_1(r) = \frac{a^2\left(\frac{\partial p}{\partial z}\right)}{\eta\pi}\left\{\left(\frac{r}{a}\right)^2\left[\ln\left(\frac{r}{a}\right)+\frac{k^4}{(1-k^4)}\ln k\right] - \frac{k^4\ln k}{(1-k^4)}\left(\frac{a}{r}\right)^2\right\} \quad (8)$$

while for $\alpha = 3/4$ the function $W_3(r)$ in equation (7) has to be substituted by

$$W_3(r) = \frac{a^2\left(\frac{\partial p}{\partial z}\right)}{3\eta\pi}\left\{\left(\frac{r}{a}\right)^2\left[\ln\left(\frac{r}{a}\right)+\frac{k^4\ln k}{(1-k^4)}\right] - \frac{k^4\ln k}{(1-k^4)}\left(\frac{a}{r}\right)^2\right\}. \quad (9)$$

In the case of a sector cross-section, i.e. $k = 0$ ($b = 0$) the solutions are given by

$$W_{2m-1}(r) = \frac{4\left(\frac{\partial p}{\partial z}\right)a^2}{\eta\pi(2m-1)\left[4-\left(\frac{2m-1}{2\alpha}\right)^2\right]} \times\left\{\left(\frac{r}{a}\right)^2 - \left(\frac{r}{a}\right)^{(2m-1)/2\alpha}\right\} \quad \text{for } \alpha \neq 1/4, 3/4 \quad (7')$$

$$W_1(r) = \frac{a^2\left(\frac{\partial p}{\partial z}\right)}{\eta\pi}\left(\frac{r}{a}\right)^2\ln\left(\frac{r}{a}\right) \quad \text{for } \alpha = 1/4 \quad (8')$$

and

$$W_3(r) = \frac{a^2\left(\frac{\partial p}{\partial z}\right)}{3\pi\eta}\left(\frac{r}{a}\right)^2\ln\left(\frac{r}{a}\right) \quad \text{for } \alpha = 3/4. \quad (9')$$

The velocity distribution for laminar flow in a tube of *sector cross-section* is therefore (Fig. 2)

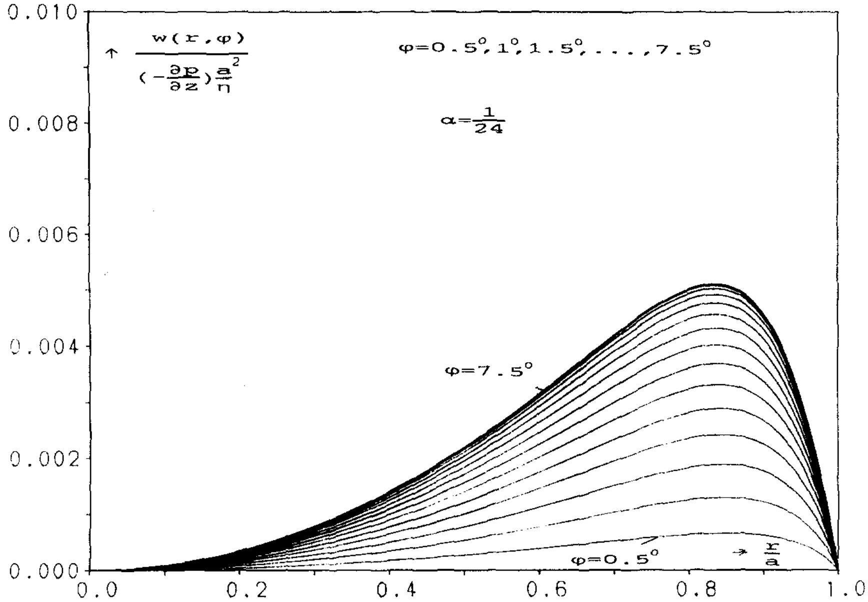

0.010
0.008
0.006
0.004
0.002
0.000
0.0
0.2
0.4
0.6
0.8
1.0
φ=0.5°, 1°, 1.5°, . . . , 7.5°
α=1/24
φ=7.5°
φ=0.5°

FIG. 2(a).

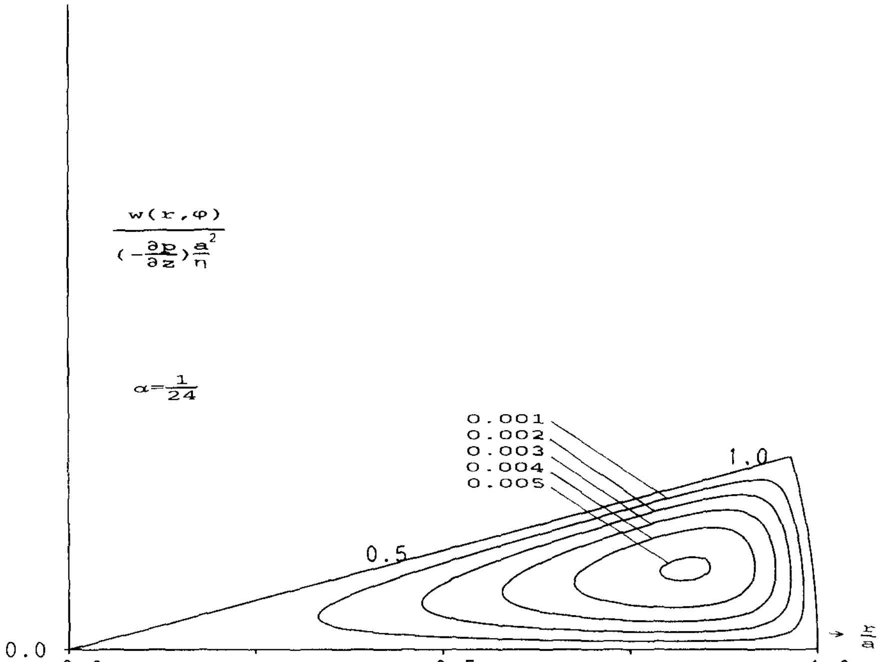

α=1/24
0.001
0.002
0.003
0.004
0.005
1.0
0.5
0.0
0.0
0.5
1.0

FIG. 2(b).

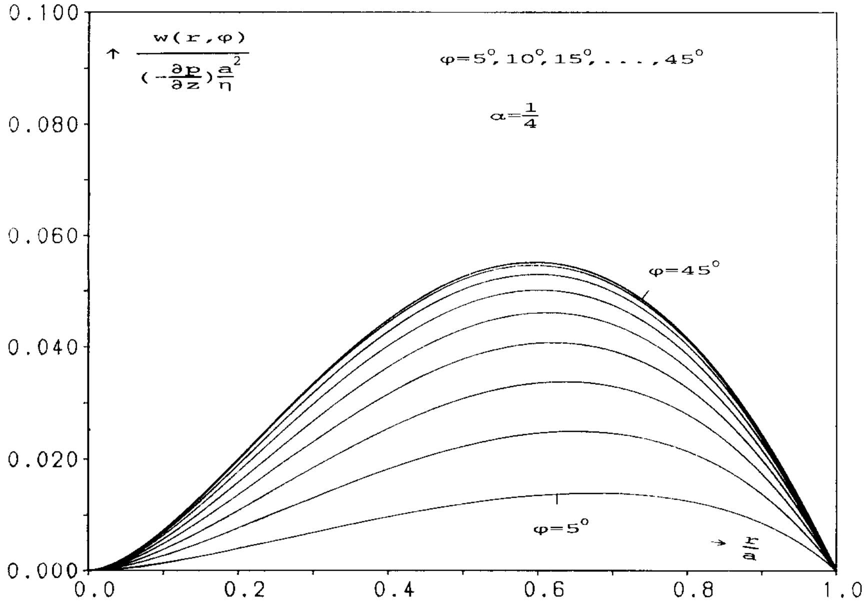

0.100
0.080
0.060
0.040
0.020
0.000
0.0
0.2
0.4
0.6
0.8
1.0
w(r,φ) / (-∂p/∂z) a²/η
φ=5°, 10°, 15°, . . . , 45°
α=1/4
φ=45°
φ=5°
r/a

FIG. 2(c).

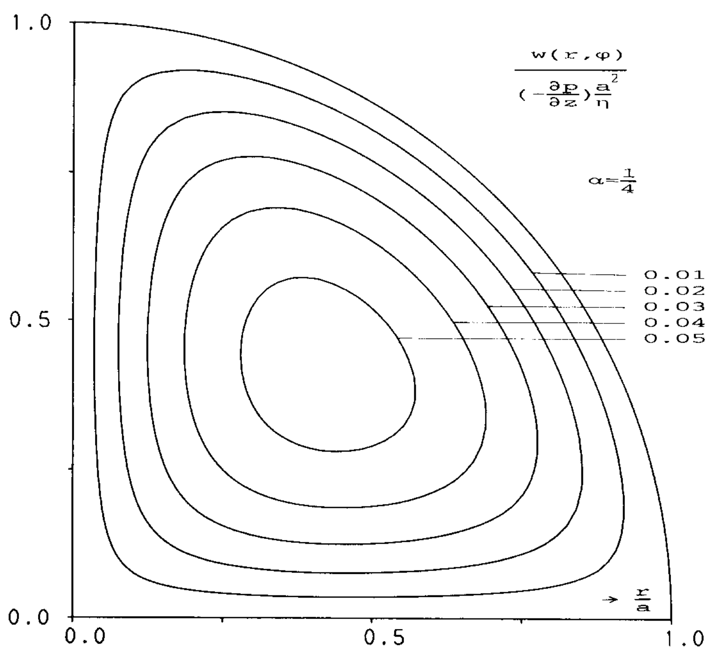

1.0
0.5
0.0
0.0
0.5
1.0
w(r,φ) / (-∂p/∂z) a²/η
α=1/4
0.01
0.02
0.03
0.04
0.05
r/a

FIG. 2(d).

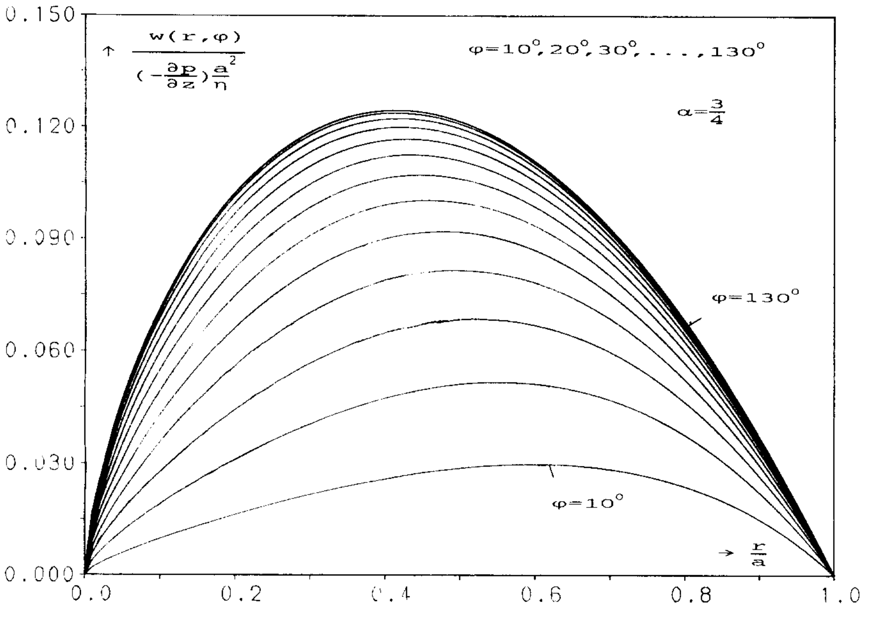


FIG. 2(e).

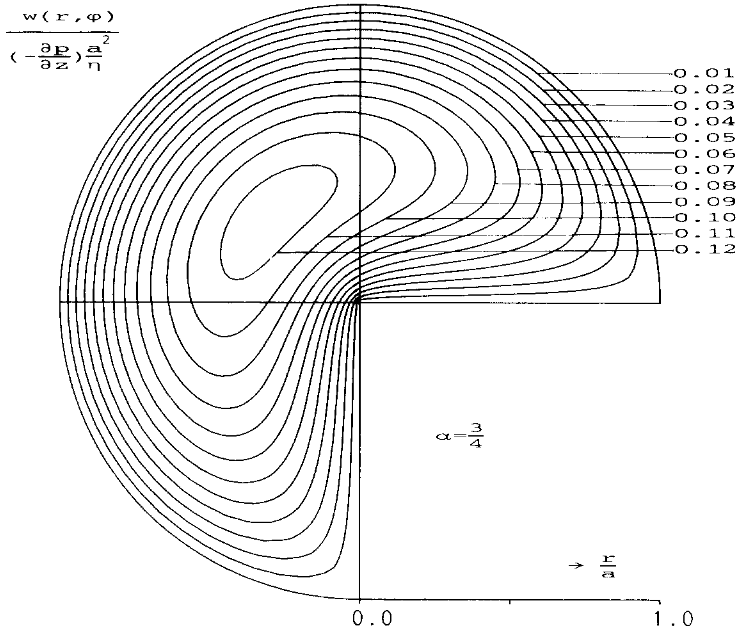


FIG. 2(f).

FIG. 2. Flow profile of laminar viscous flow and lines of equal velocity.

$$w_s(r,\phi) = \frac{4a^2\left(\frac{\partial p}{\partial z}\right)}{\pi\eta}\sum_{m=1}^{\infty}\frac{\sin\left[\frac{(2m-1)}{2\alpha}\phi\right]}{(2m-1)\left[4-\left(\frac{2m-1}{2\alpha}\right)^2\right]} \times \left\{\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^{(2m-1)/2\alpha}\right\} \quad (10')$$

where for a circular quarter cross-section $\alpha = 1/4$ the term $m = 1$ in the series has to be substituted by equation (8′) and where for a circular three-quarter cross-section $\alpha = 3/4$ the term $m = 2$ must be replaced by equation (9′). The velocity distribution for an *annular sector cross-section* is (Fig. 2)

$$w_a(r,\phi) = \frac{4a^2\left(\frac{\partial p}{\partial z}\right)}{\pi\eta}\sum_{m=1}^{\infty}\frac{\sin\left[\frac{(2m-1)}{2\alpha}\phi\right]}{(2m-1)\left[4-\left(\frac{2m-1}{2\alpha}\right)^2\right]} \times \left\{\left(\frac{r}{a}\right)^2-\frac{(1-k^{(2m-1)/2\alpha+2})}{(1-k^{(2m-1)/\alpha})}\left(\frac{r}{a}\right)^{(2m-1)/2\alpha} - \frac{k^{(2m-1)/2\alpha}(k^2-k^{(2m-1)/2\alpha})}{(1-k^{(2m-1)/\alpha})}\left(\frac{a}{r}\right)^{(2m-1)/2\alpha}\right\}. \quad (10)$$

For $\alpha = 1/4$ the term $m = 1$ has to be replaced by equation (8) and for $\alpha = 3/4$ the term $m = 2$ has to be substituted by equation (9). The flow volume per time unit is given by

$$\dot{V} = \int_0^{2\pi\alpha}\int_b^a w(r,\phi) r\,dr\,d\phi$$

and yields ($\alpha \neq 1/4, 3/4$) the expression

$$\dot{V}_s = \frac{-16a^4\alpha}{\pi\eta}\left(\frac{\partial p}{\partial z}\right)\sum_{m=1}^{\infty}\frac{\left[\frac{1}{4}-\frac{1}{((2m-1)/2\alpha)+2}\right]}{(2m-1)^2\left[4-\left(\frac{2m-1}{2\alpha}\right)^2\right]} \quad (11)$$

for the sector cross-section, and

$$\dot{V}_a = \frac{16a^4\alpha}{\pi\eta}\left(\frac{\partial p}{\partial z}\right)\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2\left[4-\left(\frac{2m-1}{2\alpha}\right)^2\right]} \times \left\{\frac{(1-k^4)}{4}-\frac{(1-k^{(2m-1)/2\alpha+2})^2}{(1-k^{(2m-1)/\alpha})\left(\frac{2m-1}{2\alpha}+2\right)} + \frac{k^{(2m-1)/\alpha}(k^2-k^{(2m-1)/2\alpha})(1-k^{2-(2m-1)/2\alpha})}{(1-k^{(2m-1)/\alpha})\left(\frac{2m-1}{2\alpha}-2\right)}\right\} \quad (12)$$

for the annular sector cross-section. For the sector angles $\alpha = 1/4$ and $3/4$ one has to replace in equation (11) the term $m = 1$ by $-a^4(\partial p/\partial z)/16\eta\pi$ and $m = 2$ by $-a^4(\partial p/\partial z)/48\pi\eta$, respectively. For the annular pipeline with the sector of $\alpha = 1/4$ and $3/4$ the term $m = 1$ has to be replaced in equation (12) by

$$-\frac{a^4(\partial p/\partial z)}{16\eta\pi}\left\{1-k^4-\frac{16k^4(\ln k)^2}{1-k^4}\right\}$$

and $m = 2$ by

$$-\frac{a^4(\partial p/\partial z)}{48\eta\pi}\left\{1-k^4-\frac{16k^6(\ln k)^2}{1-k^4}\right\}$$

respectively.

2.2. *The mass transport problem*

Since the medium is flowing the local concentration change must be determined by the effects of convection and molecular diffusion. The mass transport equation therefore reads

$$D\left[\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}+\frac{1}{r^2}\frac{\partial^2 c}{\partial \phi^2}+\frac{\partial^2 c}{\partial z^2}\right]-w(r,\phi)\frac{\partial c}{\partial z}=0. \quad (13)$$

In this equation the diffusion in the axial direction ($\sim\partial^2 c/\partial z^2$) may be neglected compared to the convective part ($\sim\partial c/\partial z$). If the flow in the tube is considered a plug flow, i.e. $w = w_0 =$ const. the partial differential equation for the mass transport is given by

$$\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}+\frac{1}{r^2}\frac{\partial^2 c}{\partial \phi^2}-\frac{w_0}{D}\frac{\partial c}{\partial z}=0$$

and has to be solved with the boundary conditions of constant wall concentration

$c = c_w$ at the walls $r = a, b$ and $\phi = 0, 2\pi\alpha$.

If the flow is laminar one has to introduce instead of w_0 expression (10), where the velocity distribution is represented as an infinite series. For both flow cases the local concentration and mean concentration is determined if at the inlet $z = 0$ the concentration $c = c_i =$ const.

3. METHOD OF SOLUTION

Two cases of mass transport in a tube of circular annular sector cross-section will be distinguished, one being that of plug flow $w = w_0$ in the tube and the other of laminar flow, the solution of which requires the knowledge of the case with plug flow.

3.1. *Plug flow*

With the dimensionless coordinate $y = r/a$ the differential equation to be solved for plug flow yields

$$\frac{\partial^2 c}{\partial y^2}+\frac{1}{y}\frac{\partial c}{\partial y}+\frac{1}{y^2}\frac{\partial^2 c}{\partial \phi^2}-\frac{w_0 a^2}{D}\frac{\partial c}{\partial z}=0 \quad (14)$$

which has to be solved with the boundary conditions ($k = b/a$)

$$c = c_w \quad \text{at the walls } y = k, 1 \text{ and } \phi = 0, 2\pi\alpha.$$

Substituting

$$\frac{c-c_w}{c_i-c_w} = C\,e^{-\lambda z}$$

results in the boundary conditions $C = 0$ at $y = k, 1$ and $\phi = 0, 2\pi\alpha$, the initial condition $C = 1$ at $z = 0$ and

$$\frac{\partial^2 C}{\partial y^2} + \frac{1}{y}\frac{\partial C}{\partial y} + \frac{1}{y^2}\frac{\partial^2 C}{\partial \phi^2} + \beta^2 C = 0$$

where

$$\beta^2 = \frac{w_0 a^2 \lambda}{D}.$$

The solution is given by

$$C(y,\phi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\{J_{m/2\alpha}(\beta_{mn}y)Y_{m/2\alpha}(\beta_{mn}) - J_{m/2\alpha}(\beta_{mn})Y_{m/2\alpha}(\beta_{mn}y)\}\sin(m/2\alpha)(\phi)$$

where β_{mn} are the roots of

$$\begin{vmatrix} J_{m/2\alpha}(\beta) & Y_{m/2\alpha}(\beta) \\ J_{m/2\alpha}(k\beta) & Y_{m/2\alpha}(k\beta) \end{vmatrix} = 0 \tag{15}$$

and A_{mn} are integration constants to be determined from the initial condition $C = 1$ at the inlet $z = 0$. They are

$$A_{mn} = \left(\int_0^{2\pi\alpha}\int_k^1 y C_{m/2\alpha}(\beta_{mn}y)\sin\left(\frac{m}{2\alpha}\phi\right)\mathrm{d}y\,\mathrm{d}\phi \Big/ \int_0^{2\pi\alpha}\int_k^1 y C^2_{m/2\alpha}(\beta_{mn}y)\sin^2\left(\frac{m}{2\alpha}\phi\right)\mathrm{d}y\,\mathrm{d}\phi\right)$$

where

$$C_{m/2\alpha}(\beta_{mn}y) \equiv J_{m/2\alpha}(\beta_{mn}y)Y_{m/2\alpha}(\beta_{mn}) - J_{m/2\alpha}(\beta_{mn})Y_{m/2\alpha}(\beta_{mn}y).$$

It is finally

$$A_{2m-1n} = \left(4\int_k^1 y C_{(2m-1)/2\alpha}(\beta_{2m-1n}y)\,\mathrm{d}y \Big/ \pi(2m-1)\int_k^1 y C^2_{(2m-1)/2\alpha}(\beta_{2m-1n}y)\,\mathrm{d}y\right). \tag{16}$$

With the orthogonality condition of $C_{m/2\alpha}(\beta_{mn}y)$, given by

$$\int_k^1 y C_{m/2\alpha}(\beta_{mn}y)C_{m/2\alpha}(\beta_{mp}y)\,\mathrm{d}y = \begin{cases} 0 & \text{for } n \neq p \\ \frac{1}{2}\{C'^2_{m/2\alpha}(\beta_{mn}) - k^2 C'^2_{m/2\alpha}(\beta_{mn}k)\} & \text{for } n = p \end{cases} \tag{17}$$

one obtains

$$A_{2m-1n} = \left(8\int_k^1 y C_{(2m-1)/2\alpha}(\beta_{2m-1n}y)\,\mathrm{d}y \Big/ \pi(2m-1) \times \{C'^2_{(2m-1)/2\alpha}(\beta_{2m-1n}) - k^2 C'^2_{(2m-1)/2\alpha}(\beta_{2m-1n}k)\}\right).$$

The local concentration is therefore given by

$$c(r,\phi,z) = c_w + (c_i - c_w)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{2m-1n} \times C_{(2m-1)/2\alpha}\left(\beta_{2m-1n}\frac{r}{a}\right)\sin\left(\frac{(2m-1)}{2\alpha}\phi\right) \times \exp\left[-\frac{D\beta^2_{2m-1n}}{w_0 a}\left(\frac{z}{a}\right)\right]. \tag{18}$$

For a sectorial cross-section ($k = 0$) the local concentration is given by the expression

$$c(r,\phi,z) = c_w + (c_i - c_w)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{2m-1n} \times J_{(2m-1)/2\alpha}\left(\bar{\beta}_{2m-1n}\frac{r}{a}\right)\sin\left(\frac{(2m-1)}{2\alpha}\phi\right) \times \exp\left[-\frac{D\bar{\beta}^2_{2m-1n}}{w_0 a}\left(\frac{z}{a}\right)\right] \tag{19}$$

where

$$A_{2m-1n} = \left(8\int_0^1 y J_{(2m-1)/2\alpha}(\bar{\beta}_{2m-1n}y)\,\mathrm{d}y \Big/ \pi(2m-1)J'^2_{(2m-1)/2\alpha}(\bar{\beta}_{2m-1n})\right) \tag{20}$$

and $\bar{\beta}_{mn}$ are roots of

$$J_{m/2\alpha}(\bar{\beta}) = 0. \tag{21}$$

From the above results all special cases may be obtained.

3.2. *Laminar flow*

The concentration for laminar flow in a tube of annular sector cross-section is obtained from the solution of the partial differential equation

$$\frac{\partial^2 c}{\partial y^2} + \frac{1}{y}\frac{\partial c}{\partial y} + \frac{1}{y^2}\frac{\partial^2 c}{\partial \phi^2} - \frac{4a^4}{\pi D\eta} \times \sum_{m=1}^{\infty}\left(\frac{\partial p}{\partial z}\right)\frac{\sin\left[\frac{(2m-1)}{2\alpha}\phi\right]}{(2m-1)\left[4-\left(\frac{2m-1}{2\alpha}\right)^2\right]} \times \left\{y^2 - \frac{(1-k^{(2m-1)/2\alpha+2})}{(1-k^{(2m-1)/\alpha})}y^{(2m-1)/2\alpha} - \frac{k^{(2m-1)/2\alpha}(k^2-k^{(2m-1)/2\alpha})}{(1-k^{(2m-1)/\alpha})}y^{-(2m-1)/2\alpha}\right\}\frac{\partial c}{\partial z} = 0 \tag{22}$$

which in the case of a tube of sector cross-section reads

$$\frac{\partial^2 c}{\partial y^2}+\frac{1}{y}\frac{\partial c}{\partial y}+\frac{1}{y^2}\frac{\partial^2 c}{\partial \phi^2}$$

$$-\frac{4a^4(\partial p/\partial z)}{\pi\eta D}\sum_{m=1}^{\infty}\frac{\sin\left[\frac{(2m-1)}{2\alpha}\phi\right]}{(2m-1)\left[4-\left(\frac{2m-1}{2\alpha}\right)^2\right]}$$

$$\times\{y^2-y^{(2m-1)/2\alpha}\}\frac{\partial c}{\partial z}=0. \quad (23)$$

The solution for the local concentration is given by equations (18) and (19), respectively, for which new eigenvalues β^*_{mn} have to be determined by the following procedure. The solution of the above treated case of plug flow is used. It exhibits the eigenvalues β_{mn}. For the above differential equation for the concentration with laminar flow a solution is assumed satisfying the same boundary conditions and having the same eigenfunctions as the case of plug flow. Thus it is with

$$\frac{c-c_w}{c_i-c_w}=C^*\,e^{-\Lambda z}$$

$$E[C^*;y,\phi]=\frac{\partial^2 C^*}{\partial y^2}+\frac{1}{y}\frac{\partial C^*}{\partial y}+\frac{1}{y^2}\frac{\partial^2 C^*}{\partial \phi^2}+\frac{4a^4\Lambda}{\pi\eta D}$$

$$\times\sum_{l=1}^{\infty}\left(\frac{\partial p}{\partial z}\right)\frac{\sin\left[\frac{(2l-1)}{2\alpha}\phi\right]}{(2l-1)\left[4-\left(\frac{2l-1}{2\alpha}\right)^2\right]}$$

$$\times\{y^2-[(1-k^{(2l-1)/2\alpha+2})y^{(2l-1)/2\alpha}+k^{(2l-1)/2\alpha}$$

$$\times(k^2-k^{(2l-1)/2\alpha})y^{-(2l-1)/2\alpha}]/(1-k^{(2l-1)/\alpha})\}C^*=0 \quad (24)$$

where the solution may be written in the form

$$C^*(y,\phi)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}C_{m/2\alpha}(\beta_{mn}y)\sin\left(\frac{m}{2\alpha}\phi\right). \quad (25)$$

This solution satisfies all boundary conditions. Introducing it into the above differential equation and observing equation (24), one obtains by employing the orthogonality relation of $C_{m/2\alpha}(\beta_{mn}y)$ after averaging a homogeneous system of algebraic equations for the approximate determination of the eigenvalues $\beta^*_{mn}(\Lambda_{mn})$, where

$$\beta^{*2}_{mn}=4a^4\Lambda_{mn}/\pi\eta D(\partial p/\partial z).$$

The remaining constants B_{mn} are obtained from the initial conditions. The Galerkin condition is

$$\int_0^{2\pi\alpha}\int_k^1 E[C^*;y,\phi]yC_{\mu/2\alpha}(\beta_{\mu\nu}y)\sin\left(\frac{\mu}{2\alpha}\phi\right)\mathrm{d}y\,\mathrm{d}\phi=0 \quad (26)$$

for $\mu = 1, 2, \ldots$ and $\nu = 1, 2, \ldots$, and yields an ∞^2 number of homogeneous infinite algebraic equations in the remaining constants B_{mn} [7]. The vanishing coefficient determinant of this system gives finally the eigenvalues β^*_{mn}. Truncating the infinite algebraic system by a finite m and n renders a determinant of finite order, which yields approximate values for the lower eigenvalues β^*_{mn}. It was found, that the computing effort of this method became more time consuming than the numerical solution of the mass transport equation. For this reason the numerical evaluation of the above mentioned determinant, as obtained by an analytical treatment of the problem was abandoned in favour of the pure numerical solution.

4. NUMERICAL EVALUATION

Some of the previous results have been evaluated numerically. The velocity distribution of laminar flow has been presented in Figs. 2(a)–(f) for various sector lines. The ratio

$$w(r,\phi)\bigg/\left(\frac{\partial p}{\partial z}\right)\frac{a^2}{\eta}$$

is shown for various values of α and coordinates r/a and ϕ. Figure 2(a) represents the velocity of the liquid in a pipeline with the sector angle of $15^\circ = 2\pi\alpha$ ($\alpha = 1/24$). Since the velocity is symmetric to $\phi = 7.5^\circ$ it is only presented from $\phi = 0$ to 7.5° as a function of r/a. It may be noted that the velocity increases with the angular coordinate ϕ and with the radius r. It reaches a maximum value close to $r = a$, of which the maximum shifts with decreasing ϕ towards the wall $r = a$. This is true in the opposite sense for $7.5^\circ \leqslant \phi \leqslant 15^\circ = 2\pi\alpha$. Figure 2(b) shows in addition the lines of equal velocity, expressing, the location of the larger velocities. Similar results are presented for $\alpha = 1/12$, i.e. $2\pi\alpha = 30^\circ$, $\alpha = 1/72$, $\alpha = (1/8)(2\pi\alpha = 45^\circ)$ in ref. [8], Figs. 2(c) and (d) and $\alpha = 1/4$, i.e. a pipeline of quarter cross-section ($2\pi\alpha = 90^\circ$). Another case of $\alpha = 3/4$, i.e. a pipeline of three-quarter cross-section ($2\pi\alpha = 270^\circ$) is shown in Figs. 2(e) and (f). Here the velocity distribution is presented for the angular angles $\phi = 10^\circ, 20^\circ, 30^\circ, \ldots, 130^\circ$. Figure 2(f) exhibits the lines of equal velocity, from which it can be noted that the maximum velocity appears in the second quadrant. The mass transport for plug flow is exhibited in Figs. 3 and 4. In Figs. 3(a)–(e) the concentration ratio $(c-c_w)/(c_i-c_w)$ is presented along the tube, expressed by the coordinate $(D/w_0a)(z/a)$. The results are given for various radii ratios $b/a = k$, sector angles $2\pi\alpha$ and angular angles ϕ for $k < r/a < 1$. Figure 3(a) shows the concentration for $k = 0.1$ and $2\pi\alpha = 30^\circ$ ($\alpha = 1/12$). First of all one detects that the concentration decreases along the length of the tube z and that it exhibits larger magnitude towards the wall $r = a$. With increasing diffusion parameter D/w_0a it decreases in magnitude, i.e. increasing diffusion coefficient D or decreasing plug flow velocity. The largest concentration profile appears in the plane of symmetry $\phi = \pi\alpha$, which is here $\phi = 15^\circ$. The con-

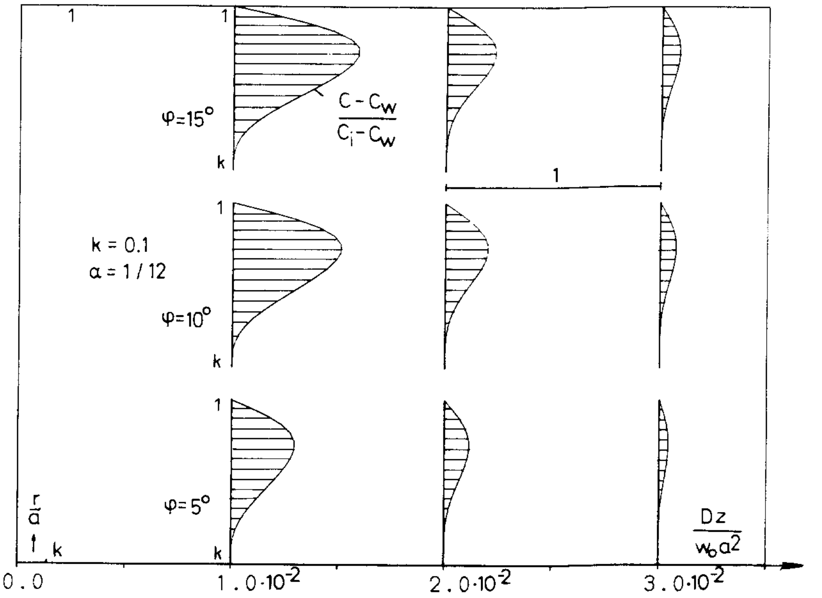

1
1
φ=15°
k
C - Cw / Ci - Cw
1
1
k = 0.1
α = 1 / 12
φ=10°
k
1
φ= 5°
r/a
k
k
Dz / w0a2
0.0
1.0·10-2
2.0·10-2
3.0·10-2

FIG. 3(a).

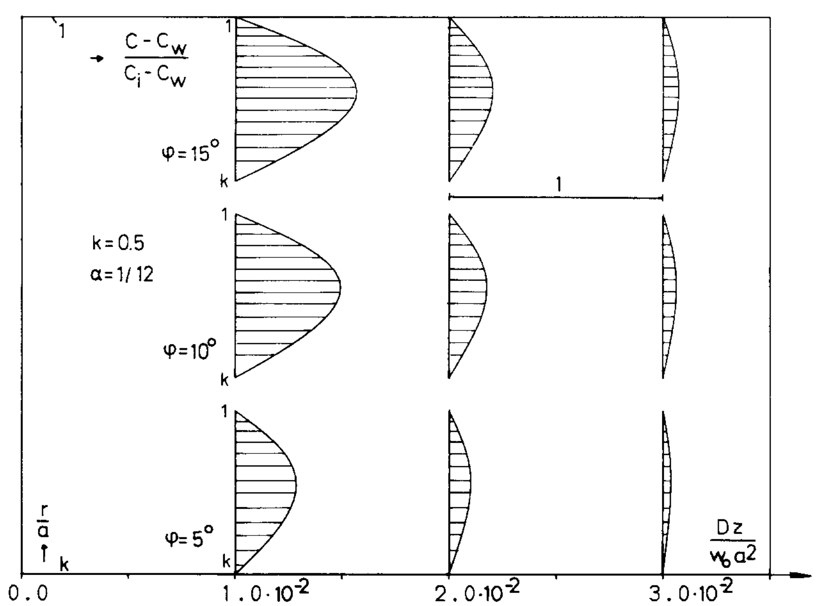

1
C - Cw / Ci - Cw
1
φ= 15°
k
1
1
k=0.5
α=1/ 12
φ=10°
k
1
r/a
φ= 5°
k
k
Dz / w0a2
0.0
1.0·10-2
2.0·10-2
3.0·10-2

FIG. 3(b).

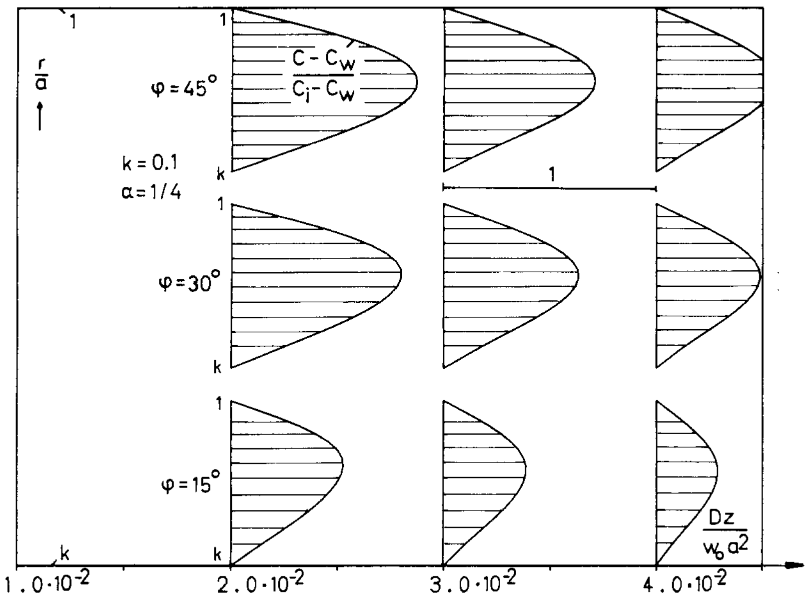

r/a
1
k
k = 0.1
α = 1/4
φ = 45°
φ = 30°
φ = 15°
(C - C_w)/(C_i - C_w)
1
Dz/(w_0 a^2)
1.0·10^-2
2.0·10^-2
3.0·10^-2
4.0·10^-2

FIG. (3c).

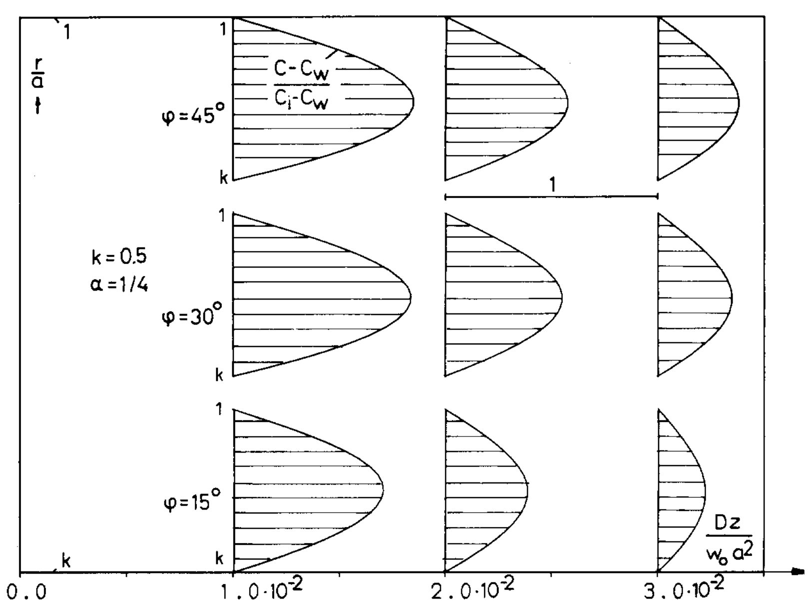

r/a
1
k
k = 0.5
α = 1/4
φ = 45°
φ = 30°
φ = 15°
(C - C_w)/(C_i - C_w)
1
Dz/(w_0 a^2)
0.0
1.0·10^-2
2.0·10^-2
3.0·10^-2

FIG. (3d).

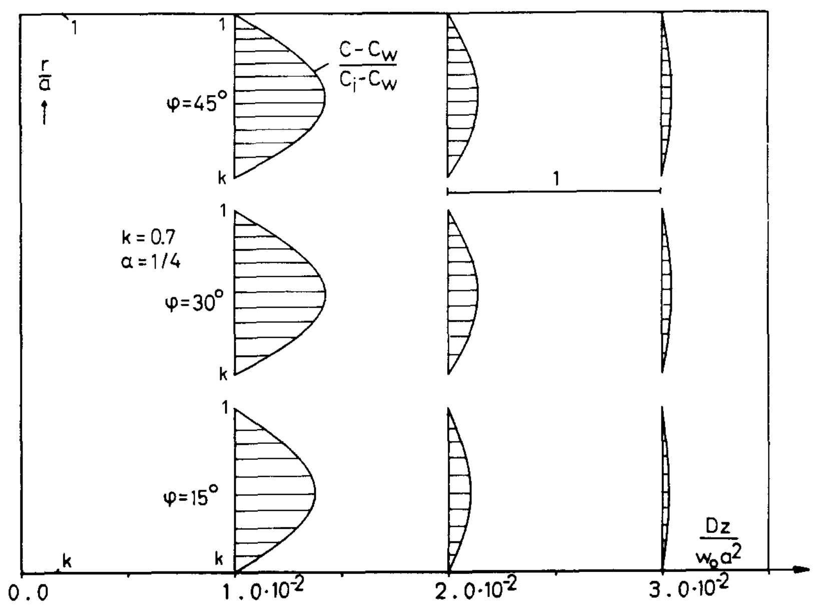


FIG. (3e).

FIG. 3. Concentration profile for plug flow along tube length.

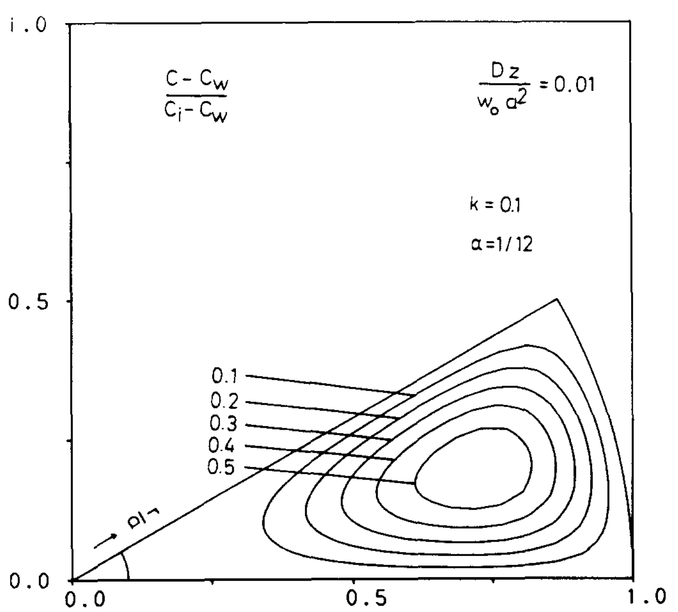
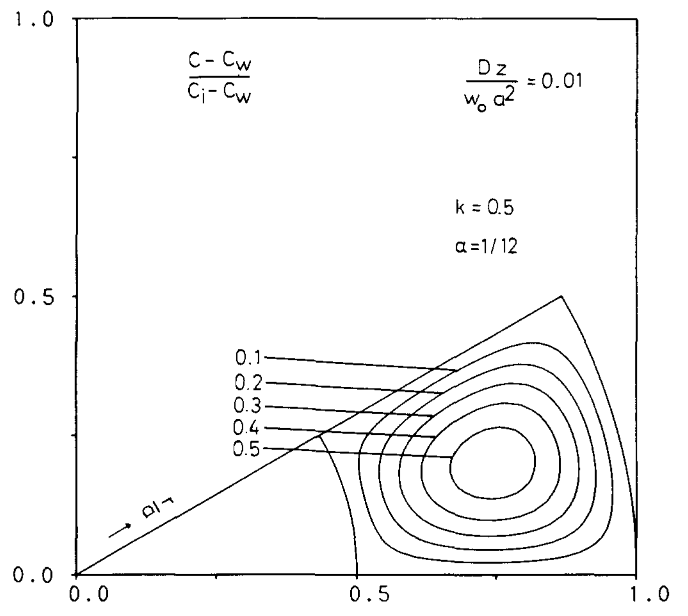
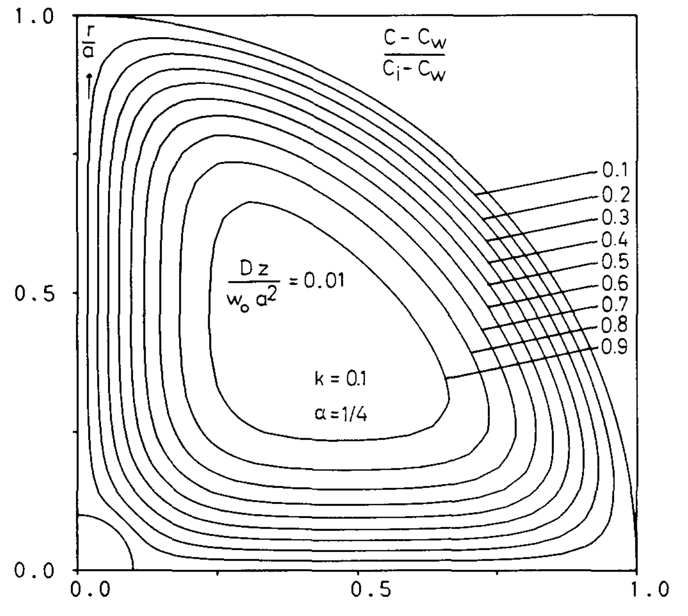


FIG. 4(a).



C - Cw / Ci - Cw
Dz / wo a2 = 0.01
k = 0.5
α = 1/12
0.1
0.2
0.3
0.4
0.5
r/a
1.0
0.5
0.0
0.0
0.5
1.0

FIG. 4(b).



C - Cw / Ci - Cw
Dz / wo a2 = 0.01
k = 0.1
α = 1/4
r/a
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0.5
0.0
0.0
0.5
1.0

FIG. 4(c).

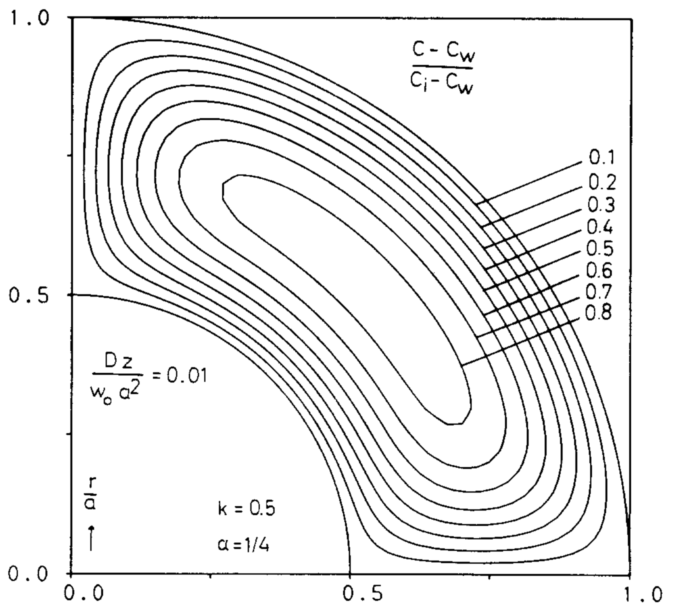


FIG. 4(d).

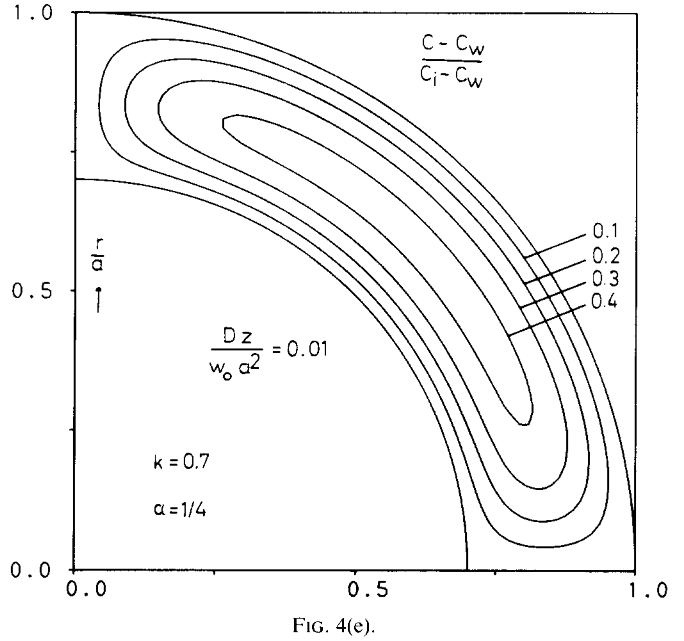


FIG. 4(e).

FIG. 4. Lines of equal concentration for plug flow.

centration for the annular sector tube of $2\pi\alpha = 30°$ and $k = 0.5$ is shown in ref. [8]. It may be mentioned that unity for the concentration ratio $(c-c_w)/(c_i-c_w)$ is shown on all figures. Similar results are given in ref. [8] for $k = 0.1$ and $\alpha = 1/6$ and $k = 0.5$ and $\alpha = 1/6$. For a quarter tube $\alpha = 1/4$, $k = 0.1$, 0.5 and 0.7 the concentration ratio for plug flow, is presented in Figs. 3(c)–(e). One can note, that with the increase of sectorial angle α the concentration decay becomes much slower along the tube, which, of course, is reduced again by changing the annular sector geometry of the cross-section, i.e. by decreasing the value of k. Figures 4(a)–(e) show the lines of equal concentration in the tube at the location $(D/w_0a)(z/a) = 0.01$ for various values of α and k. With increasing k the lines of equal concentration become more oval as the sector angle increases.

For laminar flow the distribution of the con-

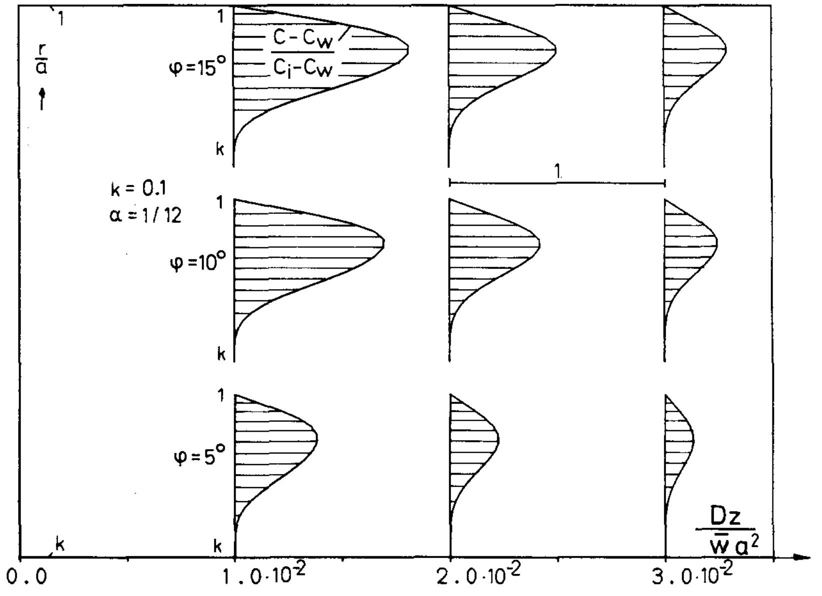


FIG. 5(a).

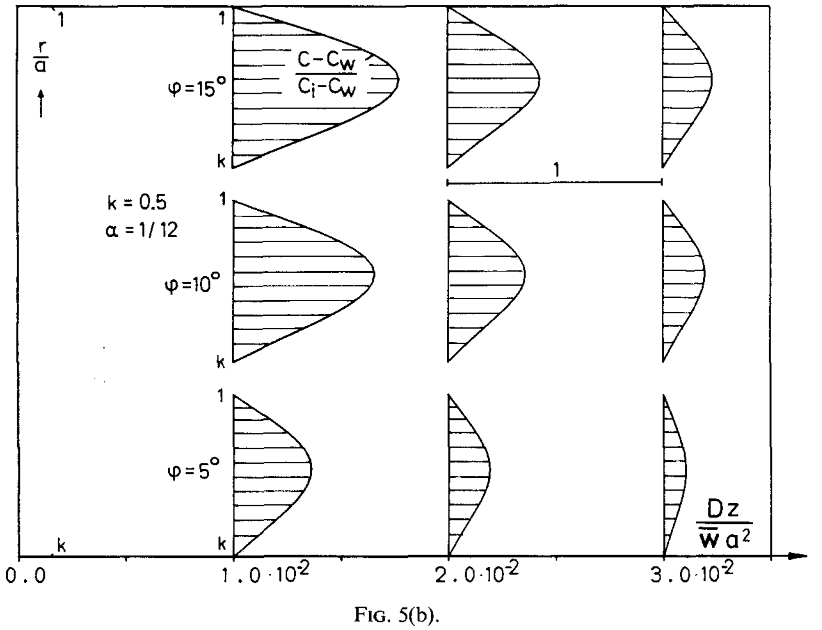


FIG. 5(b).

centration ratio $(c-c_w)/(c_i-c_w)$ is presented for various annular parameters k and sector angles $2\pi\alpha$ in Figs. 5 and 6, where Fig. 5 renders the concentration profiles along the pipe length and its cross-section, while Fig. 6 shows again the lines of equal concentration. In Figs. 5(a)–(e) the concentration is given across the cross-section of the tube and at various locations $Dz/\bar{w}a^2$ along the tube. Here $\bar{w}$ is the mean velocity of the liquid in the tube. The concentration profiles exhibit similar behaviour as in the cases of plug flow. It may, however, be noted that its magnitude is reduced less rapidly in laminar flow (compare Figs. 3 and 5). The lines of equal concentration are presented in Figs. 6(a)–(e) for various tube cross-sections (k and α) at the location $Dz/\bar{w}a^2 = 0.01$, where the larger concentration in comparison with plug flow may be noticed in the 'centre' of the tube. It may be mentioned, that the solution based on the

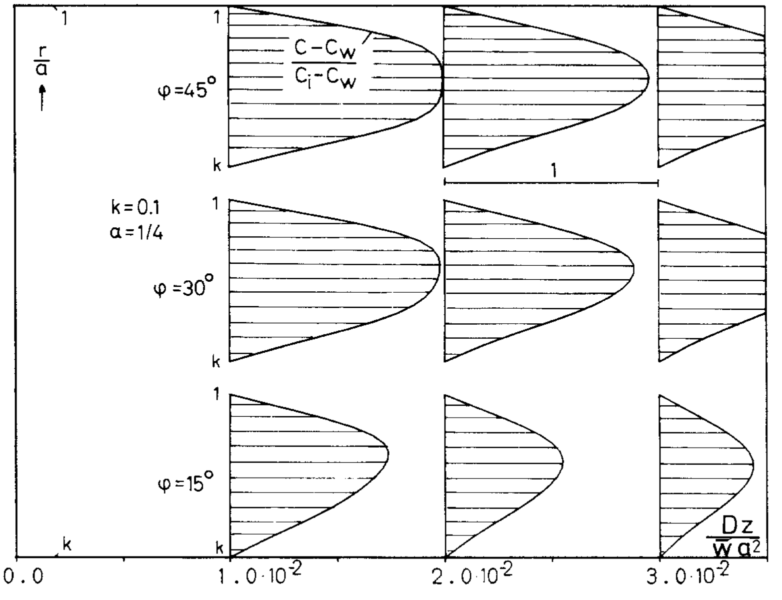


FIG. 5(c).

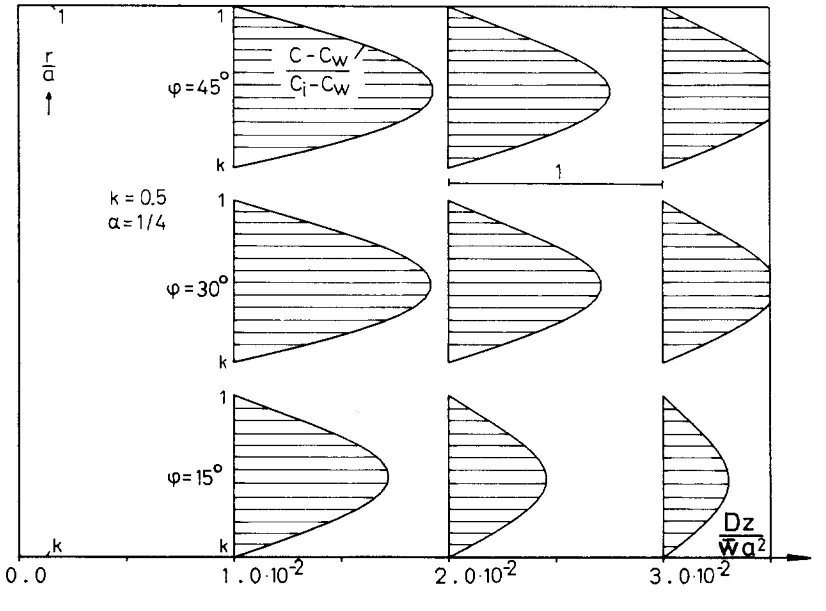


FIG. 5(d).

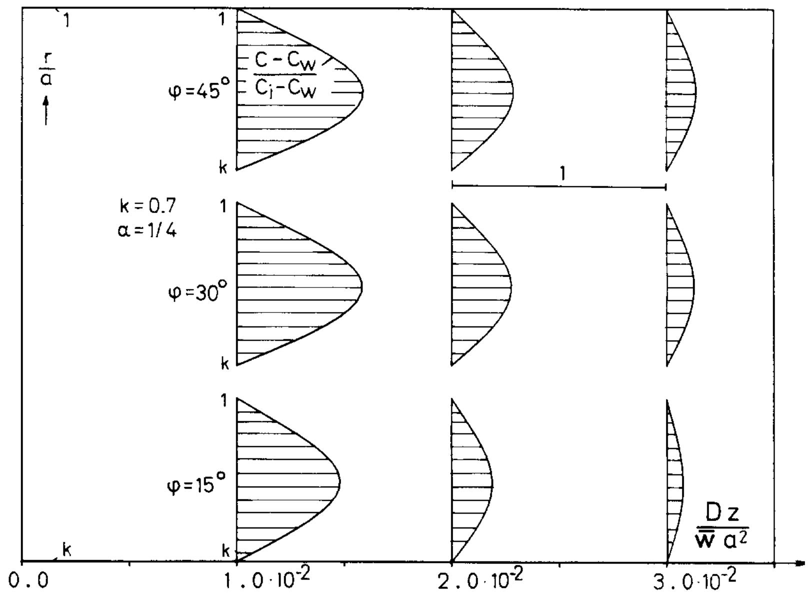


FIG. 5(e).

FIG. 5. Concentration profile for laminar viscous flow.

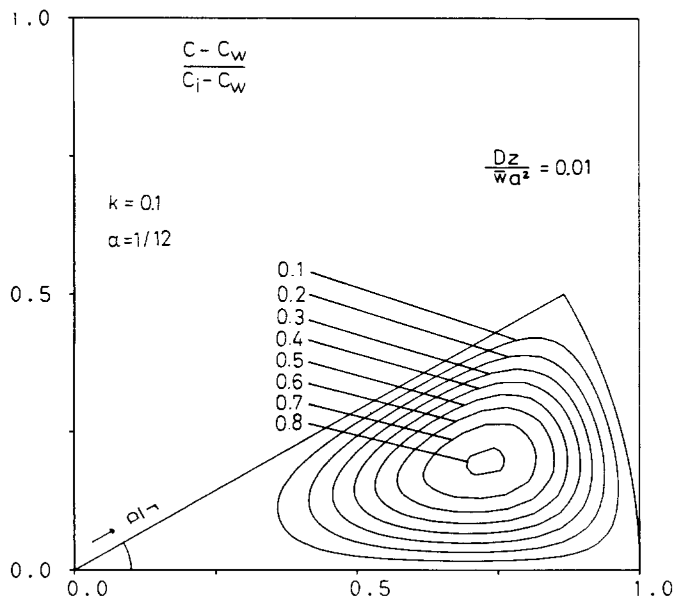
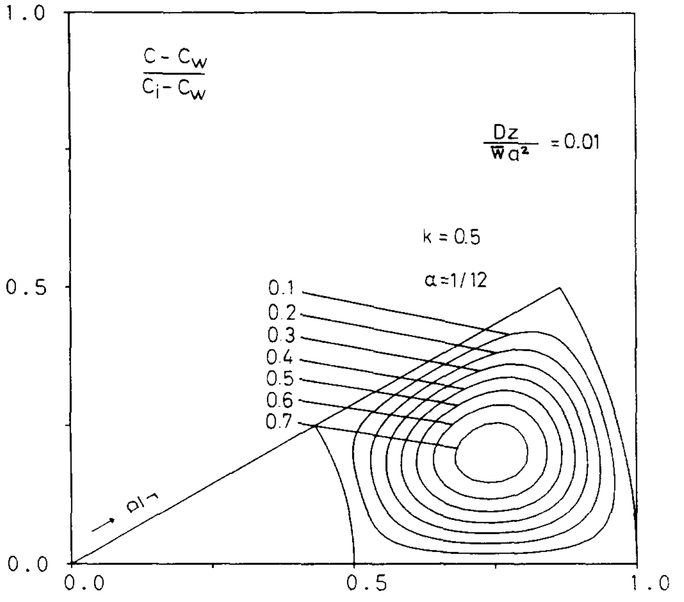
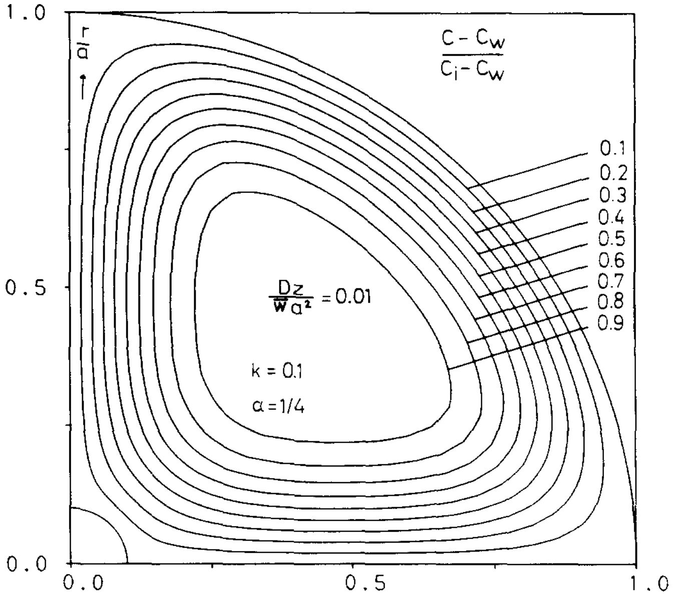


FIG. 6(a).



1.0
0.5
0.0
0.0
0.5
1.0
C - Cw / Ci - Cw
Dz / W̄a² = 0.01
k = 0.5
α = 1/12
0.1
0.2
0.3
0.4
0.5
0.6
0.7
r/a

FIG. 6(b).



1.0
0.5
0.0
0.0
0.5
1.0
r/a
C - Cw / Ci - Cw
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Dz / W̄a² = 0.01
k = 0.1
α = 1/4

FIG. 6(c).

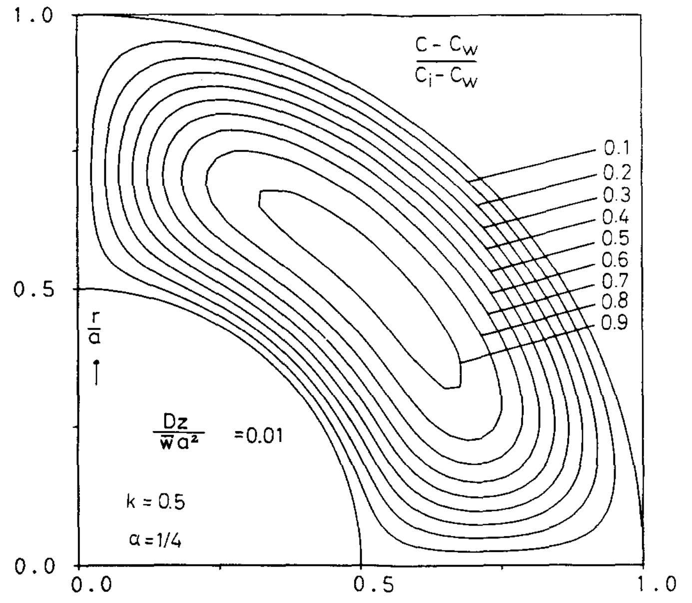


Fig. 6(d).

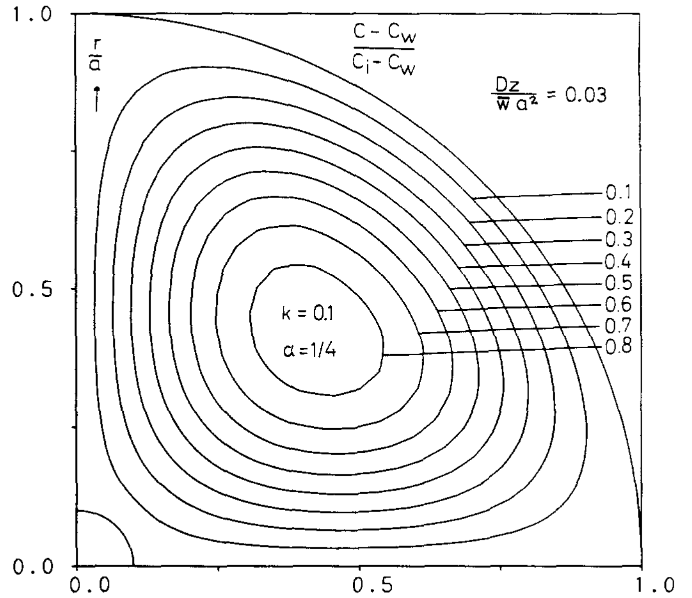


Fig. 6(e).

Fig. 6. Lines of equal concentration for laminar flow.

analytical treatment and the results of the numerical solution of the mass transport equation yield identical results.

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TRANSPORT DE MASSE DANS DES TUBES SECTEUR ET SECTEUR ANNULAIRE

Résumé—Le profil de vitesse et le transport de masse pour un écoulement piston et laminaire de liquide sont déterminés dans des tubes secteur ou secteur annulaire. Les concentrations à la paroi et à l'entrée du tube sont supposées constantes. Les profils de concentration sont déterminés en plusieurs points de la section droite et le long du tube. On présente aussi les courbes isovitesses et d'égale concentration pour les cas considérés.

STOFFÜBERGANG IN SEKTOR- UND RINGSEKTOR-ROHREN

Zusammenfassung—Für Pfropfen- und Laminar-Strömung von Flüssigkeit in Sektor- und Ringsektor-Rohren werden Geschwindigkeitsverteilung und Stoffübergang bestimmt. Die Konzentration an der Wand und die Anfangskonzentration am Rohreinlaß werden als konstant angenommen. Es werden Konzentrationsprofile für verschiedene Punkte des Querschnitts und in Längsrichtung der Rohre ermittelt. Zusätzlich werden Linien gleicher Geschwindigkeit und Konzentration der Flüssigkeit für den Fall der Pfropfen- und Laminarströmung gezeigt.

МАССОПЕРЕНОС В ТРУБАХ, РАЗДЕЛЕННЫХ НА СЕКТОРЫ, И В СЕКЦИОНИРОВАННЫХ КОЛЬЦЕВЫХ ТРУБАХ

Аннотация—Распределение скорости и массоперенос получены для стержневого и ламинарного режимов течения жидкости в трубах и кольцевых каналах, разделенных на секторы. Концентрация на стенке и начальная концентрация на входе в трубу полагались постоянными. Профили концентрации определялись в различных местах поперечного сечения и по длине трубы. Представлены графики равной скорости жидкости и равной концентрации для случаев стержневого и ламинарного режимов течения.